

## SCALING LIMITS OF DISCRETE HOLOMORPHIC FUNCTIONS

GUANGBIN REN AND ZEPING ZHU

ABSTRACT. One of the most natural and challenging issues in discrete complex analysis is to prove the convergence of discrete holomorphic functions to their continuous counterparts. This article is to solve the open problem in the general setting. To this end we introduce new concepts of *discrete surface measure* and *discrete outer normal vector* and establish the discrete Cauchy-Pompeiu integral formula,

$$f(\zeta) = \int_{\partial B^h} \mathcal{K}^h(z, \zeta) f(z) dS^h(z) + \int_{B^h} E^h(\zeta - z) \partial_z^h f(z) dV^h(z),$$

which results in the uniform convergence of the scaling limits of discrete holomorphic functions up to second order derivatives in the standard square lattices.

## 1. INTRODUCTION

Discrete complex analysis aims to find a kind of mathematical theory on lattices similar to its continuous counterpart. A typical problem in discrete complex analysis is the convergence of the scaling limits of discrete holomorphic functions.

A convergence problem for a certain kind of discrete holomorphic functions has been studied by Smirnov and his collaborators and it is of eminent importance in proving the conjecture about the conformal invariance in the Ising model [10, 5, 17]. Skopenkov [15] considered a different convergence problem and he proved that the Dirichlet boundary value problem for the real part of a discrete analytic function has a unique solution and this solution uniformly converges to a harmonic function in the case of orthogonal lattices. This issue was also investigated by Courant-Friedrichs-Lewy [6] for square lattices, and by Chelkak-Smirnov [4] for rhombic lattices. Nevertheless, the convergence of the scaling limits of discrete holomorphic functions in the general case is still an open problem.

---

2010 *Mathematics Subject Classification.* 30G25, 39A12, 49M25, 31C20.

*Key words and phrases.* Discrete complex analysis, Discrete holomorphic function, Green's theorem, Cauchy-Pompeiu formula, Scaling limit.

This work was supported by the NNSF of China (11071230).

The goal of this article is to solve this open problem in the case of the standard square lattices. This will depend heavily on our discrete Cauchy-Pompeiu integral formula

$$f(\zeta) = \int_{\partial B^h} \mathcal{K}^h(z, \zeta) f(z) dS^h(z) + \int_{B^h} E^h(\zeta - z) \partial_{\bar{z}}^h f(z) dV^h(z),$$

which clearly respects its continuous version.

To this end, a new version of the integral theory in the discrete complex analysis is developed on square lattices in the article. The classical one was initiated by Isaacs [11] and Ferrand [8], and further developed by Duffin [7], Zeilberger [20], and others. This function theory has been generalized to the cases of more complicated graphs by Mercat [13] and Bobenko-Mercat-Suris [2]. Moreover, discrete complex analysis has found its applications in fields ranging from combination geometry [14, 16], numerical analysis [9], computer graphics [1, 19], and statistical physics [18]. However, the theory does not fully mature even on square lattices.

Our new theory has several advantages over the old ones. Firstly, it distinguishes itself by the elegant discrete Cauchy-Pompeiu integral formula, which is totally analogous to its continuous version. Therefore it has good performance in applications. Next, as opposed to discrete Clifford theory where maps from  $\mathbb{Z}_h^2$  to  $\mathbb{R}^{16}$  have to be considered [3], we can consider maps from  $\mathbb{Z}_h^2$  to  $\mathbb{C}$  as expected without raising the dimension of the target domain. Finally, we can prove the uniform convergence of discrete holomorphic function up to second order derivatives.

It is worth pointing out that although our results are stated only in the complex plane, our approach is applicable to higher dimensional spaces.

This paper is organized as follows: In Section 2, we introduce some basic concepts such as *discrete surface measure* and *discrete outer normal vector*. In Sections 3 and 4 we establish the integral theory of discrete holomorphic functions; in particular, we obtain the discrete Cauchy-Pompeiu formula. In Section 5 and 6, we study the convergence problem of discrete holomorphic functions and solve the open problem as stated in Theorems 5.2, 5.3, 5.4 and 6.1. Appendix in Section 7 contains all the technical results.

## 2. DISCRETIZATION OF OPERATORS, SURFACE MEASURE, AND NORMAL VECTORS

Basic elements in discrete complex analysis are introduced in this section.

**2.1. Discrete  $\bar{\partial}$ -operator.** We will work on the discrete lattices  $\mathbb{Z}_h^2$ , a discretization of  $\mathbb{R}^2$ . Here  $\mathbb{Z}_h = h\mathbb{Z}$  for any given positive parameter  $h$ . We shall study discrete holomorphic functions related to the symmetric discretization of the classical  $\bar{\partial}$ -operator.

**Definition 2.1.** The discrete  $\bar{\partial}$ -operator and its conjugate are defined respectively as

$$\partial_{\bar{z}}^h := \frac{1}{2}(\partial_1^h + i\partial_2^h), \quad \partial_z^h := \frac{1}{2}(\partial_1^h - i\partial_2^h),$$

where  $\partial_i^h$  ( $i = 1, 2$ ) are symmetric difference operator

$$\partial_i^h = \frac{1}{2}(\partial_i^{+,h} + \partial_i^{-,h}).$$

Here  $\partial_i^{+,h}$  and  $\partial_i^{-,h}$  stand for the forward and backward difference operators respectively, i.e.,

$$\begin{aligned} \partial_i^{+,h} f(x) &= \frac{f(x + he_i) - f(x)}{h}, \\ \partial_i^{-,h} f(x) &= \frac{f(x) - f(x - he_i)}{h}, \end{aligned}$$

where  $\{e_1, e_2\}$  is the standard basis of  $\mathbb{R}^2$ .

Notice that the symmetric discretization  $\partial_{\bar{z}}^h$  converges to the classical differential operator  $\partial_{\bar{z}}$  as  $h$  tends to zero.

It is worth noting that  $\partial_{\bar{z}}^h f$  makes sense on  $B$  only for those functions  $f$  whose definition domain contain  $\overline{B}$ , the discrete closure of  $B$ .

**Definition 2.2.** Let  $B$  be a subset of  $\mathbb{Z}_h^2$ . We define its discrete closure and interior respectively as

$$\overline{B} := B \cup \partial B, \quad B^\circ := B \setminus \partial B,$$

where  $\partial B$  is the *discrete boundary* of  $B$  consisting of every point  $z \in \mathbb{Z}_h^2$  whose neighborhood

$$N(z) := \{z, z \pm h, z \pm hi\}$$

has some point inside  $B$  and some other point outside  $B$ , i.e.,

$$\partial B := \left\{ z \in \mathbb{Z}_h^2 : N(z) \cap B \neq \emptyset \quad \text{and} \quad N(z) \setminus B \neq \emptyset \right\}.$$

Now we can introduce the concept of discrete holomorphic functions.

**Definition 2.3.** A function  $f : \overline{B} \rightarrow \mathbb{C}$  with  $B \subset \mathbb{Z}_h^2$  is said to be discrete holomorphic on  $B$  if for any  $z \in B$  we have  $\partial_{\bar{z}}^h f(z) = 0$ .

**2.2. Discrete surface measure and discrete normal vector.** The concepts of discrete surface measure and discrete outer normal vector are essential to our theory.

**Definition 2.4.** Let  $B$  be a subset of  $\mathbb{Z}_h^2$ . The discrete boundary measure  $S$  on  $\partial B$  is defined as

$$S(U) = \sum_{z \in U} s(z), \quad \forall U \subset \partial B,$$

where  $s : \partial B \rightarrow \mathbb{R}$  is the density function

$$(2.1) \quad s = \frac{h^2}{2} \sqrt{\sum_{i=1}^2 \left( (\partial_i^{+,h} \chi_B)^2 + (\partial_i^{-,h} \chi_B)^2 \right)},$$

and  $\chi_B$  denotes the characteristic function of  $B$ .

**Remark 2.5.** If we extend the density function  $s$  to the whole  $\mathbb{Z}_h^2$  by zero extension, then (2.1) holds true on  $\mathbb{Z}_h^2$  since

$$\sum_{i=1,2} \left( (\partial_i^{+,h} \chi_B)^2 + (\partial_i^{-,h} \chi_B)^2 \right) = 0 \quad \text{on} \quad \mathbb{Z}_h^2 \setminus \partial B.$$

**Definition 2.6.** The discrete outer normal vector at a boundary point of  $B \subset \mathbb{Z}_h^2$  is a vector

$$\vec{n} = (n_1^+, n_1^-, n_2^+, n_2^-),$$

defined by

$$n_l^\pm = \frac{-2 \partial_l^{\pm,h} \chi_B}{\sqrt{\sum_{i=1}^2 \left( (\partial_i^{+,h} \chi_B)^2 + (\partial_i^{-,h} \chi_B)^2 \right)}}, \quad l = 1, 2.$$

It is evident that the Euclidean norm of  $\vec{n}$  is always equal to 2 on  $\partial B$ .

3. GREEN'S FORMULA ON  $\mathbb{Z}_h^2$ 

With the concepts of the discrete surface measure  $S$  and the discrete outer normal vector  $\vec{n}$  above, we can now establish the discrete version of Green's formula in this section.

Let  $V^h$  be the Haar measure on the group  $\mathbb{Z}_h^2$ . Then for any  $f : \mathbb{Z}_h^2 \rightarrow \mathbb{C}$  we have

$$\int_{\mathbb{Z}_h^2} f dV^h = \sum_{z \in \mathbb{Z}_h^2} f(z) h^2.$$

As usual, for any  $B \subset \mathbb{Z}_h^2$ ,

$$\int_B f dV^h := \int_{\mathbb{Z}_h^2} f \chi_B dV^h.$$

**Theorem 3.1** (Green's formula). *Let  $B$  be a bounded subset of  $\mathbb{Z}_h^2$ . For any function  $f : \overline{B} \rightarrow \mathbb{R}$ , we have*

$$\int_{\partial B} f n_i^\pm dS = \int_B \partial_i^{\mp, h} f dV^h, \quad (i = 1, 2).$$

In the discrete setting, we shall always extend  $S$ ,  $s$ , and  $n_i^\pm$  to the whole lattice  $\mathbb{Z}_h^2$ . That is, we identify the discrete boundary measure  $S$  with  $S \circ \varsigma^{-1}$  via the natural embedding map  $\varsigma : \partial B \rightarrow \mathbb{Z}_h^2$  and identify the density function  $s$  and the discrete normal vectors  $n_i^\pm$  with their zero extensions respectively. Notice that the density function of  $S \circ \varsigma^{-1}$  is exactly the zero extension of  $s$ .

We shall state our results in the language of distributions. To this end, we define the space of discrete test functions by

$$\mathcal{D}(\mathbb{Z}_h^2) := \{f : \mathbb{Z}_h^2 \rightarrow \mathbb{R} \mid \text{supp } f \text{ is bounded}\}$$

and the discrete distribution space by

$$\mathcal{D}^*(\mathbb{Z}_h^2) := \text{Hom}(\mathcal{D}(\mathbb{Z}_h^2), \mathbb{R}).$$

It is well-known that  $\mathcal{D}^*(\mathbb{Z}_h^2)$  contains every function  $f : \mathbb{Z}_h^2 \rightarrow \mathbb{R}$  regarded as a discrete distribution

$$\Lambda_f(g) := \int_{\mathbb{Z}_h^2} f g dV^h, \quad \forall g \in \mathcal{D}(\mathbb{Z}_h^2).$$

Furthermore, it also contains every locally finite measure  $\mu$  on the topological group  $\mathbb{Z}_h^2$  since it can be identified with a discrete distribution

$$\Lambda_\mu(g) := \int_{\mathbb{Z}_h^2} g d\mu, \quad \forall g \in \mathcal{D}(\mathbb{Z}_h^2).$$

**Lemma 3.2** (Discrete Stokes equations). *Let  $B$  be a subset of  $\mathbb{Z}_h^2$ ,  $S$  the surface measure on  $\partial B$ , and  $\vec{n} = (n_1^+, n_1^-, n_2^+, n_2^-)$  the normal vector on  $\partial B$ . Then in the sense of distribution we have*

$$(3.1) \quad -\partial_i^{\pm, h} \chi_B = n_i^\pm S, \quad (i = 1, 2)$$

$$(3.2) \quad 4\chi_{\partial B} = \sum_{i=1,2} (n_i^+)^2 + (n_i^-)^2.$$

*Proof.* First we check the second identity. By definition, we have

$$\sum_{i=1,2} (n_i^+)^2 + (n_i^-)^2 \Big|_{\partial B} \equiv 4.$$

Since  $n_i^\pm (i = 1, 2)$  vanish on  $\mathbb{Z}_h^2 \setminus B$ , we have

$$\sum_{i=1,2} (n_i^+)^2 + (n_i^-)^2 \Big|_{\mathbb{Z}_h^2 \setminus B} \equiv 0.$$

Thus (3.2) holds true.

Next we check identity (3.1). In  $\mathcal{D}^*(\mathbb{Z}_h^2)$ , we have

$$(3.3) \quad S = h^{-2} s,$$

since  $s$  is the density function of  $S$ . Indeed, for any  $f \in D((Z)_h^2)$  we have

$$\begin{aligned} \langle S, f \rangle &= \int_{\mathbb{Z}_h^2} f d(S \circ \varsigma^{-1}) = \int_{\partial B} f dS = \sum_{z \in \partial B} f(z) s(z) \\ &= \sum_{z \in \mathbb{Z}_h^2} f(z) s(z) \quad (\text{zero extension}) \\ &= \int_{\mathbb{Z}_h^2} h^{-2} s f dV^h \\ &= \langle h^{-2} s, f \rangle \end{aligned}$$

as desired. So we need to show that

$$-\partial_i^{\pm, h} \chi_B = h^{-2} n_i^\pm s.$$

For any  $x \in \partial B$ , it follows from Definitions 2.4 and 2.6 that

$$\begin{aligned} n_i^\pm(x)s(x) &= \frac{-2\partial_i^{\pm,h}\chi_B}{\sqrt{\sum_{i=1,2}(\partial_i^{+,h}\chi_B)^2 + (\partial_i^{-,h}\chi_B)^2}} \frac{h^2}{2} \sqrt{\sum_{i=1,2}(\partial_i^{+,h}\chi_B)^2 + (\partial_i^{-,h}\chi_B)^2} \\ &= -h^2\partial_i^{\pm,h}\chi_B. \end{aligned}$$

On the other hand, for any  $x \notin \partial B$  we have

$$s(x) = \partial_i^{\pm,h}\chi_B(x) = 0.$$

Hence

$$-\partial_i^{\pm,h}\chi_B = h^{-2}n_i^\pm s.$$

This completes the proof.  $\square$

**Remark 3.3.** In the continuous setting the Green theorem

$$\int_{\partial B} f n_i dS = \int_B \frac{\partial f}{\partial x_i} dV$$

can be restated in term of distribution as

$$(3.4) \quad n_i S = -\frac{\partial}{\partial x_i} \chi_B, \quad (i = 1, 2).$$

Here  $n_i$  is identified with its zero extension from  $\partial B$  to  $\mathbb{R}^2$  and the surface measure  $S$  is identified with its push-out  $S \circ \varsigma^{-1}$  via the classical embedding map  $\varsigma$  from  $\partial B$  to  $\mathbb{R}^2$ . Moreover, in this point of view we have

$$(3.5) \quad \sum_{i=1}^2 n_i^2 = \chi_{\partial B}.$$

The system of discrete Stokes equations in Lemma 3.1 is a variant of its continuous counterparts in (3.4) and (3.5). However, the discrete normal vectors is likely to diverge under scaling limits.

We point out that the discrete surface measure  $S$  and the discrete normal vector  $\vec{n}$  are uniquely determined by the system of Stokes equations in Lemma 3.2. Its proof is the same as in the continuous version as we shall see in the next lemma.

**Lemma 3.4.** *Let  $B$  be a domain in  $\mathbb{C}$  with smooth boundary  $\partial B$ . Then the system of the Stokes equations*

$$(3.6) \quad m_i T = -\frac{\partial}{\partial x_i} \chi_B, \quad (i = 1, 2)$$

$$(3.7) \quad \sum_{i=1}^2 m_i^2 = \chi_{\partial B},$$

with  $T$  being a non-negative regular Borel measure on  $\partial B$ , and  $\vec{m} = (m_1, m_2)$  smooth on  $\partial B$ , has a unique solution  $\{T, \vec{m}\} = \{S, \vec{n}\}$ , where  $S$  is the boundary measure and  $\vec{n}$  the outer normal vector on  $\partial B$ .

*Proof.* We only need to prove the uniqueness since  $\{S, \vec{n}\}$  is clearly a solution.

Assume  $\{T, \vec{m}\}$  is another solution. For any non-negative  $f \in C_0^\infty(\mathbb{C})$ ,

$$\int_{\partial B} f dT = \int_{\partial B} f m_1^2 dT + \int_{\partial B} f m_2^2 dT.$$

Since  $\{S, \vec{n}\}$  is also a solution, then we have  $n_i S = m_i T$  ( $i = 1, 2$ ). It follows that

$$\begin{aligned} \int_{\partial B} f dT &= \int_{\partial B} f m_1^2 dT + \int_{\partial B} f m_2^2 dT \\ &= \int_{\partial B} f m_1 n_1 dS + \int_{\partial B} f m_2 n_2 dS, \end{aligned}$$

which implies that

$$\int_{\partial B} f dT \leq \int_{\partial B} f dS$$

since  $\vec{m}$  and  $\vec{n}$  are both unit vector and  $f$  is non-negative.

By symmetricity, we have

$$\int_{\partial B} f dS \leq \int_{\partial B} f dT.$$

This implies  $T = S$  since the set of finite combinations of non-negative smooth functions with compact support is dense in  $C_c(\partial B)$ .

Since  $n_i S = m_i T$  ( $i = 1, 2$ ) and  $S = T$ , we have  $n_i = m_i$  a.e.  $S$  on  $\partial B$ .

According to the assumption that  $\vec{m}$  and  $\vec{n}$  are both smooth, we obtain that

$$\vec{m} \equiv \vec{n}.$$

□



Now we are in position to prove our main result in this section.

Proof of Theorem 3.1. Put

$$g = f\chi_{\overline{B}}.$$

Since  $\overline{B}$  is bounded, we have  $g \in \mathcal{D}(\mathbb{Z}_h^2)$  so that (3.1) implies

$$\begin{aligned} \langle n_i^\pm S, g \rangle &= \langle -\partial_i^{\pm, h} \chi_B, g \rangle \\ &= - \int_{\mathbb{Z}_h^2} \partial_i^{\pm, h} \chi_B(x) g(x) dV^h(x). \end{aligned}$$

It is easy to verify that

$$- \int_{\mathbb{Z}_h^2} \partial_i^{\pm, h} \chi_B(x) g(x) dV^h(x) = \int_{\mathbb{Z}_h^2} \chi_B(x) \partial_i^{\mp, h} g(x) dV^h(x).$$

We thus have

$$(3.8) \quad \langle n_i^\pm S, g \rangle = \int_B \partial_i^{\mp, h} g dV^h = \int_B \partial_i^{\mp, h} f dV^h.$$

On the other hand, it follows from identity (3.3) that

$$\begin{aligned} \langle n_i^\pm S, g \rangle &= \langle h^{-2} n_i^\pm s, g \rangle = \int_{\mathbb{Z}_h^2} h^{-2} n_i^\pm s g dV^h = \sum_{z \in \mathbb{Z}_h^2} n_i^\pm(z) s(z) g(z) \\ &= \sum_{z \in \partial B} n_i^\pm(z) s(z) g(z) \quad (\text{zero extension}) \\ &= \int_{\partial B} g n_i^\pm dS. \end{aligned}$$

Since  $f = g$  on  $\partial B$ , we thus obtain

$$\langle n_i^\pm S, g \rangle = \int_{\partial B} f n_i^\pm dS,$$

This together with (3.8) leads to the Green theorem.  $\square$

#### 4. DISCRETE CAUCHY-POMPEIU INTEGRAL FORMULA

In this section we introduce the discrete Bochner-Martinelli kernel and establish the discrete Cauchy-Pompeiu integral formula.

The fundamental solution [12] of operator  $\partial_{\bar{z}}^h$  is defined by

$$E^h(x, y) = \frac{1}{h} E\left(\frac{x}{h}, \frac{y}{h}\right),$$

where

$$(4.1) \quad E(x, y) = \frac{1}{4\pi^2} \int_{[-\pi, \pi]^2} \frac{2}{i \sin u - \sin v} e^{i(ux+vy)} du dv.$$

Let  $\delta_0^h$  be the discrete Dirac delta function on  $\mathbb{Z}_h^2$ , i.e.,

$$\delta_0^h(z) = \begin{cases} h^{-2}, & z = 0, \\ 0, & z \neq 0. \end{cases}$$

**Theorem 4.1.** [12] *Function  $E^h$  is the fundamental solution of  $\partial_{\bar{z}}^h$ , i.e.,*

$$\partial_{\bar{z}}^h E^h = \delta_0^h \quad \text{in } \mathbb{Z}_h^2.$$

*Proof.* Without loss of generality, we can assume  $h = 1$ . By direct calculation,

$$\partial_{\bar{z}}^h(e^{i(ux+vy)}) = \frac{i \sin u - \sin v}{2} e^{i(ux+vy)}.$$

Applying  $\partial_{\bar{z}}^h$  on both sides of (4.1), we thus obtain

$$\partial_{\bar{z}}^h E^h = \frac{1}{4\pi^2} \int_{[-\pi, \pi]^2} e^{i(ux+vy)} dudv = \delta_0^h.$$

□

Let  $\mathcal{K}^h$  be the discrete Bochner-Martinelli kernel, defined by

$$\mathcal{K}^h(z, \zeta) = A(z - \zeta)n_1^-(z) + B(z - \zeta)n_1^+(z) + iC(z - \zeta)n_2^-(z) + iD(z - \zeta)n_2^+(z),$$

where

$$\begin{aligned} A(z) &= -4^{-1}E^h(h - z), \\ B(z) &= -4^{-1}E^h(-h - z), \\ C(z) &= -4^{-1}E^h(ih - z), \\ D(z) &= -4^{-1}E^h(-ih - z). \end{aligned}$$

**Theorem 4.2** (Cauchy-Pompeiu). *Let  $B$  be a bounded subset of  $\mathbb{Z}_h^2$ . Then for any function  $f : B \rightarrow \mathbb{C}$  we have*

$$(4.2) \quad \chi_B(\zeta)f(\zeta) = \int_{\partial B} \mathcal{K}^h(z, \zeta)f(z)dS(z) + \int_B E^h(\zeta - z)\partial_{\bar{z}}^h f(z)dV^h(z).$$

*Proof.* We first split the first summand in the right side of (4.2) into four parts:

$$\begin{aligned}
 \int_{\partial B} \mathcal{K}^h(z, \zeta) f(z) dS(z) &= \int_{\partial B} A(z - \zeta) f(z) n_1^-(z) dS(z) + \\
 &\quad \int_{\partial B} B(z - \zeta) f(z) n_1^+(z) dS(z) + \\
 (4.3) \quad &\quad i \int_{\partial B} C(z - \zeta) f(z) n_2^-(z) dS(z) + \\
 &\quad i \int_{\partial B} D(z - \zeta) f(z) n_2^+(z) dS(z) \\
 &= I_1 + I_2 + I_3 + I_4.
 \end{aligned}$$

By definition, we have

$$A = -4^{-1} \tau_x^{-1} \rho E^h, \quad B = -4^{-1} \tau_x \rho E^h, \quad C = -4^{-1} \tau_y^{-1} \rho E^h, \quad D = -4^{-1} \tau_y \rho E^h,$$

where the operator  $\rho$  is the reflection

$$\rho f(z) = f(-z),$$

and  $\tau_x, \tau_y$  are translations

$$\tau_x f(z) = f(z + h), \quad \tau_y f(z) = f(z + hi).$$

Applying Theorem 3.1 to  $I_1$ , we have

$$\begin{aligned}
 I_1 &= \int_B \partial_1^{+,h} [A(\cdot - \zeta) f(\cdot)](z) dV(z) \\
 &= \int_B \tau_x A(z - \zeta) \partial_1^{+,h} f(z) + \partial_1^{+,h} A(z - \zeta) f(z) dV^h(z).
 \end{aligned}$$

Similarly, we have

$$\begin{aligned}
 I_2 &= \int_B \tau_x^{-1} B(z - \zeta) \partial_1^{-,h} f(z) + \partial_1^{-,h} B(z - \zeta) f(z) dV^h(z), \\
 I_3 &= i \int_B \tau_y C(z - \zeta) \partial_2^{+,h} f(z) + \partial_2^{+,h} C(z - \zeta) f(z) dV^h(z), \\
 I_4 &= i \int_B \tau_y^{-1} D(z - \zeta) \partial_2^{-,h} f(z) + \partial_2^{-,h} D(z - \zeta) f(z) dV^h(z).
 \end{aligned}$$

Substituting the identities above to (4.3), we obtain

$$\begin{aligned}
 &\int_{\partial B} \mathcal{K}^h(z, \zeta) f(z) dS(z) \\
 &= \int_B (\tau_x A(z - \zeta) \partial_1^{+,h} f(z) + \tau_x^{-1} B(z - \zeta) \partial_1^{-,h} f(z) + i \tau_y C(z - \zeta) \partial_2^{+,h} f(z) \\
 &\quad + i \tau_y^{-1} D(z - \zeta) \partial_2^{-,h} f(z)) dV^h(z) + \int_B (\partial_1^{+,h} A(z - \zeta) + \partial_1^{-,h} B(z - \zeta) \\
 &\quad + i \partial_2^{+,h} C(z - \zeta) + i \partial_2^{-,h} D(z - \zeta)) f(z) dV^h(z).
 \end{aligned}$$

By direct calculation. we have

$$\tau_x A = \tau_x^{-1} B = \tau_y C = \tau_y^{-1} D = -4^{-1} \rho E$$

and

$$\partial_1^{+,h} A + \partial_1^{-,h} B + i\partial_2^{+,h} C + i\partial_2^{-,h} D = \rho(\partial_{\bar{z}}^h E^h) = \delta_0^h.$$

These lead to

$$\int_{\partial B} \mathcal{K}^h(z, \zeta) f(\zeta) dS(\zeta) = - \int_B E^h(\zeta - z) \partial_{\bar{z}}^h f(z) dV^h(z) + \int_B \delta_0^h(z - \zeta) f(\zeta) dV^h(z)$$

as desired.  $\square$

**Corollary 4.3.** *If  $f$  is discrete holomorphic on a bounded subset  $B$  of  $\mathbb{Z}_h^2$ , then*

$$\chi_B(\zeta) f(\zeta) = \int_{\partial B} \mathcal{K}^h(z, \zeta) f(z) dS(z).$$

**Remark 4.4.** The preceding theorem indicates a new phenomena that discrete holomorphic functions behave differently with continuous counterparts on boundaries since for any discrete holomorphic function we have

$$\int_{\partial B} \mathcal{K}^h(z, \zeta) f(\zeta) dS(\zeta) = \begin{cases} f(\zeta), & \zeta \in \partial^+ B; \\ 0, & \zeta \in \partial^- B. \end{cases}$$

where  $\partial^\pm B$  constitute a partition of  $\partial B$ , defined by

$$\partial^+ B := \partial B \cap B,$$

$$\partial^- B := \partial B \setminus B.$$

Finally, we study the holomorphicity of the Bochner-Martinelli kernel. It turns out that  $\mathcal{K}^h(z, \cdot)$  is discrete holomorphic outside the neighbourhood of the diagonal.

**Theorem 4.5.** *For any given  $z \in \partial B$ , the discrete Bochner-Martinelli kernel  $\mathcal{K}^h(z, \cdot)$  is discrete holomorphic on  $(\mathbb{Z}_h^2 \setminus \partial B) \cup (\mathbb{Z}_h^2 \setminus N(z))$ .*

We leave the proof to appendix I since its proof is direct but unpleasant.

**Remark 4.6.** To consider the holomorphicity of the kernel  $\mathcal{K}^h$  along the neighborhood of the diagonal, we denote

$$\Gamma := \{(z, \zeta) : z \in \partial^- B, \zeta \in B \cap N(z)\} \cup \{(z, \zeta) : z \in \partial^+ B, \zeta \in N(z) \setminus B\}.$$

Then one can verify from the proof of Theorem 4.5 that

$$\partial_{\zeta}^h \mathcal{K}^h(z, \zeta) = 0, \quad (z, \zeta) \notin \Gamma.$$

and when  $(z, \zeta) \in \Gamma$  we have  $\zeta \in N(z)$  and

$$\partial_{\zeta}^h \mathcal{K}^h(z, \zeta) = \begin{cases} -\frac{1}{4h^2} n_1^+(z), & \zeta = z + h, \\ -\frac{1}{4h^2} n_1^-(z), & \zeta = z - h, \\ -\frac{1}{4h^2} n_2^+(z), & \zeta = z + hi, \\ -\frac{1}{4h^2} n_2^-(z), & \zeta = z - hi. \end{cases}$$

## 5. APPROXIMATION AND CONVERGENCE

For the convergence and approximation, we shall see that a function is holomorphic if and only if it is the scaling limit of discrete holomorphic functions.

**5.1. Approximation.** For any given holomorphic function  $f$ , we come to construct discrete holomorphic functions  $f^h$  converging to  $f$ .

First, we need a concept about the convergence of discrete sets.

**Definition 5.1.** Let  $B$  be a bounded open set in  $\mathbb{C}$ . We say  $B^h \subseteq \mathbb{Z}_h^2$  converges to  $B$  and denote as

$$\lim_{h \rightarrow 0^+} B^h = B$$

if the distances between  $\partial B$  and  $\partial B^h$  as well as between  $\overline{B}$  and  $B^h$  converge to zero, i.e.,

$$\begin{aligned} \lim_{h \rightarrow 0^+} \max_{\alpha \in \partial B} \min_{\beta \in \partial B^h} \|\alpha - \beta\| &= 0, \\ \lim_{h \rightarrow 0^+} \max_{\alpha \in \partial B^h} \min_{\beta \in \partial B} \|\alpha - \beta\| &= 0, \\ \lim_{h \rightarrow 0^+} \max_{\alpha \in \overline{B}} \min_{\beta \in B^h} \|\alpha - \beta\| &= 0, \\ \lim_{h \rightarrow 0^+} \max_{\alpha \in B^h} \min_{\beta \in \overline{B}} \|\alpha - \beta\| &= 0. \end{aligned}$$

Recall for any  $\Omega \subset \mathbb{Z}_h^2$  the interior of  $\Omega$  is defined by

$$\Omega^\circ = \Omega \setminus \partial\Omega,$$

and  $S^h$  and  $V^h$  represent the discrete surface measure on  $B$  and the Haar measure in the lattice  $\mathbb{Z}_h^2$ , respectively. Let  $H(B)$  denote the space of holomorphic functions in the domain  $B \subset \mathbb{C}$ .

Now we come to the first main result in this subsection.

**Theorem 5.2.** *Let  $B$  be a bounded open set in  $\mathbb{C}$  and set*

$$B^h := (B \cap \mathbb{Z}_h^2)^\circ \subset \mathbb{Z}_h^2.$$

*If  $f \in C^3(\overline{B}) \cap H(B)$ , then the functions*

$$(5.1) \quad f^h(\zeta) := \int_{\partial B^h} \mathcal{K}^h(z, \zeta) f(z) dS^h(z)$$

*are discrete holomorphic on  $(B^h)^\circ$  and convergent to  $f$  in the sense that*

$$\lim_{h \rightarrow 0^+} \max_{B^h} |f - f^h| = 0.$$

*Proof.* According to Lemma 7.2, we know that  $B^h$  converges to  $B$ . Since  $\mathcal{K}^h(z, \cdot)$  is discrete holomorphic on  $(B^h)^\circ$  for any given  $z \in \partial B^h$ , it follows that  $f^h$  is also discrete holomorphic on  $(B^h)^\circ$ .

Now we prove that  $f^h$  converges to  $f$ . By Theorem 4.2 we have

$$f(\zeta) - f^h(\zeta) = \int_{B^h} E^h(\zeta - z) \partial_{\bar{z}}^h f(z) dV^h(z)$$

for any  $\zeta \in B^h$  so that

$$|f(\zeta) - f^h(\zeta)| \leq \max_{B^h} |\partial_{\bar{z}}^h f| \int_{B^h} |E^h(\zeta - z)| dV^h(z)$$

and by Hölder's inequality

$$|f(\zeta) - f^h(\zeta)| \leq \max_{B^h} |\partial_{\bar{z}}^h f| \left( \int_{B^h} 1 dV^h(z) \right)^{2/3} \left( \int_{B^h} |E^h(\zeta - z)|^3 dV^h(z) \right)^{1/3}.$$

Since the measure  $V^h$  is invariant under group operations of  $\mathbb{Z}_h^2$ , we have

$$\begin{aligned} \int_{B^h} |E^h(\zeta - z)|^3 dV^h(z) &\leq \int_{\mathbb{Z}_h^2} |E^h(z)|^3 dV^h(z) \\ &= \sum_{x, y \in \mathbb{Z}_h^2} \left| \frac{1}{h} E\left(\frac{x}{h}, \frac{y}{h}\right) \right|^3 h^2 \\ &= \frac{1}{h} \sum_{s, t \in \mathbb{Z}^2} |E(s, t)|^3 \\ &= \frac{1}{h} \int_{\mathbb{Z}^2} |E|^3 dV \\ &= O(h^{-1}). \end{aligned}$$

The last step used Lemma 7.5. By Lemmas 7.3 and 7.4, we have

$$\begin{aligned} \max_{B^h} |\partial_{\bar{z}}^h f| &= O(h^2), \\ \int_{B^h} 1 dV^h &= O(1). \end{aligned}$$

The above estimates thus yield

$$\max_{B^h} |f - f^h| \leq O(h^2) O(1) O(h^{-\frac{1}{3}}) = O(h^{\frac{5}{3}}).$$

This completes the proof.  $\square$

Notice that Theorem 5.2 holds true merely on the specific sets  $(B \cap \mathbb{Z}_h^2)^\circ$ . If we impose slightly strong conditions on  $f$ , we can show the convergence in more general cases.

**Theorem 5.3.** *Let  $B$  be a bounded open set in  $\mathbb{C}$  and  $f \in H(\overline{B})$ . If  $B^h$  converges to  $B$ , there exists  $f^h : B^h \rightarrow \mathbb{C}$  discrete holomorphic on  $(B^h)^\circ$  and convergent to  $f$  in the sense that*

$$\lim_{h \rightarrow 0^+} \max_{B^h \cap B} |f - f^h| = 0.$$

*Proof.* Take an open set  $U$  of  $\mathbb{C}$  such that  $f \in H(U)$  and  $\overline{B} \subset U$ . If  $B^h$  converges to  $B$ , we have  $\partial B^h \subset U$  for any  $h$  sufficiently small. Therefore we can define  $f^h$  as in (5.1). With this modification, the result follows from the same argument as in Theorem 5.2 with  $B^h$  in place of  $(B \cap \mathbb{Z}_h^2)^\circ$ .  $\square$

**5.2. Convergence.** The scaling limit of discrete holomorphic functions is shown to be holomorphic in this subsection.

**Theorem 5.4.** *Let  $B$  be a bounded open set in  $\mathbb{C}$  and  $B^h \subset \mathbb{Z}_h^2$  convergent to  $B$ . If  $f^h : B^h \rightarrow \mathbb{C}$  is discrete holomorphic on  $(B^h)^\circ$  and convergent to a function  $f \in C(B)$  in the sense that*

$$\lim_{h \rightarrow 0^+} \max_{B^h \cap B} |f - f^h| = 0,$$

*then  $f \in H(B)$ .*

Its proof relies on the following key fact.

**Proposition 5.5.** *Let  $B$  be a bounded open set in  $\mathbb{C}$  and  $B^h \subset \mathbb{Z}_h^2$  convergent to  $B$ . Then for any  $U \subset\subset B$ , there exists  $\delta > 0$  such that when  $h < \delta$ , we have*

$$U \bigcap \mathbb{Z}_h^2 \subset B^h.$$

*Proof.* Without loss of generality, we can assume that  $U = B(z_0, R) \subset\subset B$  due to compactness.

Since  $B^h$  is convergent to  $B$ , by definition we have

$$\lim_{h \rightarrow 0^+} \max_{\alpha \in \overline{B}} \min_{\beta \in B^h} \|\alpha - \beta\| = 0,$$

which means that for any given  $\epsilon > 0$

$$B^h + B(0, \epsilon) \supset \overline{B} \ni z_0,$$

provided  $h$  sufficiently small. In particular, taking  $\epsilon = R$ , we have

$$B^h + B(0, R) \ni z_0,$$

i.e.,  $(z_0 - B(0, R)) \cap B^h \neq \emptyset$ . By assumption  $U = B(z_0, R) = z_0 - B(0, R)$ , this means

$$U \bigcap B^h \neq \emptyset$$

when  $h$  is small enough.

On the other hand, we denote

$$d = \text{dist}(U, \partial B) := \inf_{\alpha \in U, \beta \in \partial B} \|\alpha - \beta\|.$$

Again the convergence of  $B^h$  implies

$$\lim_{h \rightarrow 0^+} \max_{\alpha \in \partial B^h} \min_{\beta \in \partial B} \|\alpha - \beta\| = 0.$$

For any  $h$  sufficiently small, we then have

$$\partial B^h \subset \partial B + B(0, d)$$

so that

$$\partial B^h \bigcap U \subset (\partial B + B(0, d)) \bigcap U = \emptyset$$

since  $d = \text{dist}(U, \partial B)$ .

Now we have proved that

$$U \bigcap B^h \neq \emptyset, \quad U \bigcap \partial B^h = \emptyset$$



for any  $h$  small enough. Based on these facts, we come to show that

$$U \cap \mathbb{Z}_h^2 \subset B^h, \quad \forall h < 1.$$

Assume this is not valid, then there exists  $h^* > 0$  such that

$$U \cap B^{h^*} \neq \emptyset, \quad U \cap \partial B^{h^*} = \emptyset, \quad U \cap \mathbb{Z}_{h^*}^2 \not\subset B^{h^*}.$$

Now we take two elements

$$\alpha \in U \cap B^{h^*} = U \cap \mathbb{Z}_{h^*}^2 \cap B^{h^*}, \quad \beta \in (U \cap \mathbb{Z}_{h^*}^2) \setminus B^{h^*}.$$

Since

$$\alpha, \beta \in U \cap \mathbb{Z}_{h^*}^2 = B(z_0, R) \cap \mathbb{Z}_{h^*}^2$$

and the last set is discrete connected in the square lattice  $\mathbb{Z}_{h^*}^2$ , there exist  $\{z_k\}_{k=1}^m \subset U \cap \mathbb{Z}_{h^*}^2$  such that

$$z_1 = \alpha, \quad z_m = \beta, \quad z_{k+1} \in N(z_k)$$

for any  $k = 1, 2, \dots, m-1$ . Notice that  $\alpha \in B^{h^*}$  and  $\beta \notin B^{h^*}$ , we can take  $1 \leq k^* \leq m-1$  such that

$$z_{k^*} \in B^{h^*}, \quad z_{k^*+1} \notin B^{h^*}.$$

They are both in  $\partial B^{h^*}$  since  $z_{k^*+1} \in N(z_{k^*})$ . This implies that

$$z_{k^*} \in \partial B^{h^*} \cap U,$$

which violates the assumption  $\partial B^{h^*} \cap U = \emptyset$ . This completes the proof.  $\square$

The preceding proposition results in the  $w^*$ -convergence of discrete measures, which is essential in the proof of Theorem 5.4.

**Lemma 5.6.** *Let  $B$  be a bounded open set in  $\mathbb{C}$  and  $B^h \subset \mathbb{Z}_h^2$  convergent to  $B$ .*

*Then we have*

$$\lim_{h \rightarrow 0^+} \int_{B^h \cap B} f dV^h = \int_B f dV$$

*for any  $f \in C_c(B)$ . That is,*

$$w^* - \lim_{h \rightarrow 0^+} V^h \mid_{B^h \cap B} = V \mid_B \quad \text{in} \quad (C_c(B))^*.$$

*Proof.* For any given  $f \in C_c(B)$ , we can take  $U$  to be a finite union of balls such that

$$(5.2) \quad \text{supp} f \subset U \subset\subset B.$$

In view of Proposition 5.5, for  $h$  sufficiently small we have

$$U \cap \mathbb{Z}_h^2 \subset B^h \subset \mathbb{Z}_h^2,$$

which intersects with  $U$  to yield  $B^h \cap U = U \cap \mathbb{Z}_h^2$ . This together with (5.2) concludes that

$$\int_{B^h \cap B} f dV^h = \int_{B^h \cap U} f dV^h = \int_{U \cap \mathbb{Z}_h^2} f dV^h.$$

The last integral is identical to a Riemann sum of a certain Riemann integral, which means

$$\lim_{h \rightarrow 0^+} \int_{B^h \cap B} f dV^h = \lim_{h \rightarrow 0^+} \int_{U \cap \mathbb{Z}_h^2} f dV^h = \int_U f dV = \int_B f dV.$$

□

Now we can give the proof of our main result in this subsection.

Proof of Theorem 5.4. Let  $f^h : B^h \rightarrow \mathbb{C}$  be discrete holomorphic on  $(B^h)^\circ$  and convergent to a function  $f \in C(B)$ . In order to prove  $f \in H(B)$ , we only need to verify

$$\int_B f \partial_{\bar{z}} \phi dV = 0$$

for any  $\phi \in C_0^\infty(B)$ , i.e,  $f$  is holomorphic in the sense of distribution. By Lemma 5.6,

$$\lim_{h \rightarrow 0^+} \int_{B^h \cap B} f \partial_{\bar{z}} \phi dV^h = \int_B f \partial_{\bar{z}} \phi dV.$$

It remains to prove

$$\lim_{h \rightarrow 0^+} \int_{B^h \cap B} f \partial_{\bar{z}} \phi dV^h = 0.$$

We now separate the last integral into three parts

$$(5.3) \quad \int_{B^h \cap B} f \partial_{\bar{z}} \phi dV^h = I_1^h + I_2^h + I_3^h,$$

where

$$\begin{aligned} I_1^h &= \int_{B^h \cap B} f^h \partial_{\bar{z}}^h \phi dV^h, \\ I_2^h &= \int_{B^h \cap B} (f - f^h) \partial_{\bar{z}}^h \phi dV^h, \\ I_3^h &= \int_{B^h \cap B} f (\partial_z \phi - \partial_z^h \phi) dV^h. \end{aligned}$$

For the first term  $I_1^h$ , since  $\phi \in C_0^\infty(B)$  we have

$$\text{supp} \partial_{\bar{z}}^h \phi \subset B$$

for  $h$  sufficiently small. Hence

$$I_1^h = \int_{B^h} f^h \partial_{\bar{z}}^h \phi dV^h$$

and by Theorem 3.1 we get

$$\begin{aligned} (5.4) \quad I_1^h &= \frac{1}{4} \int_{\partial B^h} \left( \phi(z+h)n_1^+(z) + \phi(z-h)n_1^-(z) + i\phi(z+ih)n_2^+(z) \right. \\ &\quad \left. + i\phi(z-ih)n_2^-(z) \right) f^h(z) dS^h(z) - \int_{B^h} \phi \partial_{\bar{z}}^h f^h dV^h. \end{aligned}$$

Pick an open set  $U$  such that

$$\text{supp} \phi \subset\subset U \subset\subset B$$

and let  $h$  be small enough obeying

$$\text{supp} \phi + B(0, 4h) \subset U.$$

By Proposition 5.5, we have

$$U \bigcap \mathbb{Z}_h^2 \subset B^h, \quad \forall h \ll 1.$$

As a result,

$$(\text{supp} \phi + B(0, 4h)) \bigcap \mathbb{Z}_h^2 \subset B^h.$$

This implies if

$$\alpha \in (\text{supp} \phi + B(0, 2h)) \bigcap \mathbb{Z}_h^2,$$

then

$$N(\alpha) \subset B(\alpha, 2h) \bigcap \mathbb{Z}_h^2 \subset B^h$$

so that

$$(\text{supp} \phi + B(0, 2h)) \bigcap \mathbb{Z}_h^2 \subset (B^h)^\circ.$$

Consequently,

$$(\text{supp}\phi + B(0, 2h)) \cap \partial B^h = \emptyset$$

which means that  $\phi$ ,  $\phi(\cdot \pm h)$  and  $\phi(\cdot \pm ih)$  all vanish on  $\partial B^h$ . It follows that

$$\int_{\partial B^h} (\phi(\cdot + h)n_1^+ + \phi(\cdot - h)n_1^- + i\phi(\cdot + ih)n_2^+ + i\phi(\cdot - ih)n_2^-) f^h dS^h = 0$$

and

$$\int_{B^h} \phi \partial_{\bar{z}}^h f^h dV^h = \int_{B^h \setminus \partial B^h} \phi \partial_{\bar{z}}^h f^h dV^h = \int_{(B^h)^\circ} \phi \partial_{\bar{z}}^h f^h dV^h = 0.$$

The last step used the fact that  $f^h$  is discrete holomorphic. Hence  $I_1^h$  vanishes for sufficiently small  $h$  according to (5.4).

Next we estimate the second item  $I_2^h$  in (5.3). By definition,

$$\begin{aligned} |I_2^h| &\leq \int_{B^h \cap B} |(f - f^h) \partial_{\bar{z}}^h \phi| dV^h \\ &\leq \max_{B^h \cap B} |f - f^h| \max_{\mathbb{R}^2} |\partial_{\bar{z}}^h \phi| \int_{B^h \cap B} 1 dV^h. \end{aligned}$$

Since  $B$  is bounded, we can assume  $B \subset B(0, R)$  for some  $R > 0$  so that

$$B^h \cap B \subset B(0, R) \cap \mathbb{Z}_h^2.$$

As shown in the proof of Lemma 5.6,

$$\lim_{h \rightarrow 0^+} \int_{B(0, R) \cap \mathbb{Z}_h^2} 1 dV^h = \int_{B(0, R)} 1 dV$$

and this means

$$\int_{B^h \cap B} 1 dV^h = O(1).$$

Since  $\phi \in C_0^\infty(B)$ , we have

$$\max_{\mathbb{R}^2} |\partial_{\bar{z}}^h \phi| = O(1).$$

In addition, according to assumption,

$$\max_{B^h \cap B} |f - f^h| = o(1).$$

The above facts together imply that  $I_2^h = o(1)$ .

Now it remains to show that

$$I_3^h = o(1).$$

By definition

$$\begin{aligned} |I_3^h| &\leq \int_{B^h \cap B} |f(\partial_{\bar{z}}\phi - \partial_{\bar{z}}^h\phi)| dV^h \\ &\leq \max_{\text{supp}\phi + B(0,h)} |f| \max_{\mathbb{R}^2} |\partial_{\bar{z}}\phi - \partial_{\bar{z}}^h\phi| \int_{B^h \cap B} 1 dV^h. \end{aligned}$$

Since  $f \in C(B)$  and  $\phi \in C_0^\infty(B)$ , we have

$$\max_{\text{supp}\phi + B(0,h)} |f| = O(1), \quad \max_{\mathbb{R}^2} |\partial_{\bar{z}}\phi - \partial_{\bar{z}}^h\phi| = o(1).$$

As we have observed,

$$\int_{B^h \cap B} 1 dV^h = O(1),$$

we thus obtain  $I_3^h = o(1)$  and this completes the proof.  $\square$

**Remark 5.7.** The convergence problem of discrete holomorphic fermions has been considered in the study of the invariance of Ising model by Smirnov [17]. His approach relies on an important fact that the discrete holomorphic fermion is a solution of the discrete Riemann boundary value problem.

## 6. UNIFORM CONVERGENCE OF DERIVATIVES

The uniform convergence of derivatives up to second order is shown for the family of the discrete holomorphic functions in this section.

Notice that if  $f^h$  is defined on  $B^h$ , then its first order derivatives can only be defined in  $(B^h)^\circ$  and similarly for its second order derivatives.

If  $B^h$  converges to  $B$ , we find  $B^h \cap B$  is almost the same as  $B \cap \mathbb{Z}_h^2$  in view of Proposition 5.5 and, by direct verification,

$$\lim_{h \rightarrow 0^+} (B^h)^\circ = B, \quad \lim_{h \rightarrow 0^+} (B^h)^{\circ\circ} = B.$$

Here  $(B^h)^{\circ\circ}$  stands for the discrete interior of  $(B^h)^\circ$ .

**Theorem 6.1.** *Let  $B$  be a bounded open set and  $f \in H(\overline{B})$ . If  $B^h$  converges to  $B$ , then there exists  $f^h : B^h \rightarrow \mathbb{C}$  discrete holomorphic on  $(B^h)^\circ$  and uniformly*

convergent to  $f$  in the sense that

$$\begin{aligned}\lim_{h \rightarrow 0^+} \max_{B^h \cap B} |f - f^h| &= 0, \\ \lim_{h \rightarrow 0^+} \max_{(B^h)^\circ \cap B} |\partial_z f - \partial_z^h f^h| &= 0, \\ \lim_{h \rightarrow 0^+} \max_{(B^h)^{\circ\circ} \cap B} |(\partial_z)^2 f - (\partial_z^h)^2 f^h| &= 0.\end{aligned}$$

*Proof.* Take an open subset  $U$  of  $\mathbb{C}$  such that  $f \in H(U)$  and  $U \supset \bar{B}$ . When  $h$  is sufficiently small, we have  $\partial B^h \subset U$  since  $B^h$  converges to  $B$ . Therefore we can define  $f^h$  as in (5.1), i.e.,

$$f^h(\zeta) = \int_{\partial B^h} \mathcal{K}^h(z, \zeta) f(z) dS^h(z).$$

The first identity in Theorem 6.1 has been shown in Theorem 5.3.

To prove the last two identities, notice that

$$\lim_{h \rightarrow 0^+} \max_B \left\{ |\partial_z f - \partial_z^h f| + |(\partial_z)^2 f - (\partial_z^h)^2 f| \right\} = 0$$

since  $\partial^h$  converges to  $\partial$  in  $C^2(B)$ . Therefore, it is sufficient to show that

$$\begin{aligned}\lim_{h \rightarrow 0^+} \max_{(B^h)^\circ \cap B} |\partial_z^h f - \partial_z^h f^h| &= 0, \\ \lim_{h \rightarrow 0^+} \max_{(B^h)^{\circ\circ} \cap B} |(\partial_z^h)^2 f - (\partial_z^h)^2 f^h| &= 0.\end{aligned}$$

According to Theorem 4.2, we have

$$f = f^h + \int_{B^h} E^h(\cdot - z) \partial_{\bar{z}}^h f(z) dV^h(z) \quad \text{on } B^h.$$

Applying difference operators on both sides, we obtain

$$\partial_z^h f - \partial_z^h f^h = \int_{B^h} \partial_z^h E^h(\cdot - z) \partial_{\bar{z}}^h f(z) dV^h(z) \quad \text{on } (B^h)^\circ$$

and

$$(\partial_z^h)^2 f - (\partial_z^h)^2 f^h = \int_{B^h} (\partial_z^h)^2 E^h(\cdot - z) \partial_{\bar{z}}^h f(z) dV^h(z) \quad \text{on } (B^h)^{\circ\circ}.$$

These together with the Hölder inequality thus imply that

$$\max_{(B^h)^\circ \cap B} |\partial_z^h f - \partial_z^h f^h| \leq \max_{B^h} |\partial_{\bar{z}} f| \left( \int_{\mathbb{Z}_h^2} (\partial_z^h E^h)^2 dV^h \right)^{1/2} \left( \int_{B^h} 1 dV^h \right)^{1/2}$$

and

$$\max_{(B^h)^{\circ\circ} \cap B} |(\partial_z^h)^2 f - (\partial_z^h)^2 f^h| \leq \max_{B^h} |\partial_{\bar{z}} f| \int_{\mathbb{Z}_h^2} |(\partial_z^h)^2 E^h| dV^h.$$

On the other hand, according to Lemmas 7.3 and 7.4, we know

$$\max_{B^h} |\partial_{\bar{z}} f| = O(h^2), \quad \int_{B^h} 1 dV^h = O(1).$$

By Lemma 7.6, we obtain

$$\int_{\mathbb{Z}_h^2} (\partial_z^h E^h)^2 dV^h = h^2 \int_{\mathbb{Z}^2} \left( \frac{1}{h^2} \partial_z^1 E^1 \right)^2 dV^1 = \frac{1}{h^2} \|\partial_z^1 E^1\|_{L^2}^2 = O\left(\frac{1}{h^2}\right)$$

and

$$\int_{\mathbb{Z}_h^2} |(\partial_z^h)^2 E^h| dV^h = h^2 \int_{\mathbb{Z}^2} \left| \frac{1}{h^3} (\partial_z^1)^2 E^1 \right| dV^1 = \frac{1}{h} \|(\partial_z^1)^2 E^1\|_{L^1} = O\left(\frac{1}{h}\right).$$

Altogether, the above results yield

$$\max_{(B^h)^\circ \cap B} |\partial_z^h f - \partial_z^h f^h| \leq O(h^2) O\left(\frac{1}{h}\right) O(1) = O(h),$$

and

$$\max_{(B^h)^{\circ\circ} \cap B} |(\partial_z^h)^2 f - (\partial_z^h)^2 f^h| \leq O(h^2) O\left(\frac{1}{h}\right) = O(h).$$

This completes the proof.  $\square$

**Remark 6.2.** Theorem 6.1 holds true under the weak condition that  $f \in C^2(\overline{B}) \cap H(B)$  provided that the specific case that  $B^h = (B \cap \mathbb{Z}_h^2)^\circ$  is considered.

## 7. APPENDIX: TECHNICAL LEMMAS

Some technical results are included in this appendix.

**Remark 7.1.** For any  $B \subset \mathbb{Z}_h^2$ , its discrete boundary has two layers:

$$\partial^+ B := \partial B \cap B,$$

$$\partial^- B := \partial B \setminus B.$$

Either both are empty or both not.

**Lemma 7.2.** *Let  $B$  be a bounded open set in  $\mathbb{C}$  and  $B^h = (B \cap \mathbb{Z}_h^2)^\circ$ . Then  $B^h$  converges to  $B$ .*

*Proof.* To prove that  $\lim_{h \rightarrow 0^+} B^h = B$ , we need to verify four conditions in Definition 5.1.

(i) Since  $B^h$  is contained in  $B$ , we have

$$\min_{\beta \in \overline{B}} \|\alpha - \beta\| = 0, \quad \forall \alpha \in B^h$$

so that

$$\lim_{h \rightarrow 0^+} \max_{\alpha \in B^h} \min_{\beta \in \overline{B}} \|\alpha - \beta\| = 0.$$

(ii) Next we want to show

$$\lim_{h \rightarrow 0^+} \max_{\alpha \in \overline{B}} \min_{\beta \in B^h} \|\alpha - \beta\| = 0,$$

i.e., for any  $\epsilon > 0$ , there exists  $\delta > 0$ , such that when  $h \leq \delta$  we have

$$(7.1) \quad \overline{B} \subset B^h + B(0, 2\epsilon).$$

To prove (7.1), we let  $\epsilon > 0$  and  $z \in \overline{B}$  be given. Since  $B$  is open, we can pick  $\delta > 0$  and  $z' \in B$  such that

$$B(z', \delta) \subset \subset B \cap B(z, \epsilon).$$

Therefore,

$$B(z', \delta) \cap \mathbb{Z}_h^2 \subset (B \cap \mathbb{Z}_h^2)^\circ = B^h$$

for any  $h$  sufficiently small. This implies

$$B(z', \delta) \cap B^h = B(z', \delta) \cap \mathbb{Z}_h^2 \neq \emptyset, \quad \forall h < 1.$$

Since  $B(z, \epsilon) \cap B^h \supset B(z', \delta) \cap B^h$ , it follows that

$$(7.2) \quad B(z, \epsilon) \cap B^h \neq \emptyset, \quad \forall h < 1.$$

Since  $\overline{B}$  is compact, one can find  $z_1, \dots, z_n$  such that

$$\overline{B} \subset \bigcup_{k=1}^n B(z_k, \epsilon).$$

As we have derived there exists  $\eta_k > 0$  such that

$$B(z_k, \epsilon) \cap B^h \neq \emptyset$$

whenever  $h < \eta_k$ . Hence when  $h < \min_{1 \leq k \leq n} \eta_k$ , we have

$$\overline{B} \subset B^h + B(0, 2\epsilon)$$

as desired.



(iii) Now we come to show

$$\lim_{h \rightarrow 0^+} \max_{\alpha \in \partial B^h} \min_{\beta \in \partial B} \|\alpha - \beta\| = 0.$$

It is sufficient to verify that

$$\min_{\beta \in \partial B} \|z - \beta\| \leq 2h, \quad \forall z \in \partial B^h.$$

Its proof is split into two cases:

(a) case 1:  $z \in \partial^+ B^h$ .

For any  $\zeta \in \partial^+(B \cap \mathbb{Z}_h^2)$ , by definition we have  $\zeta \in B$  and there is a point  $\zeta' \in \mathbb{Z}_h^2 \setminus B$  with  $\|\zeta' - \zeta\| = h$ . Since  $\zeta \in B$  and  $\zeta' \notin B$ , we obtain

$$\min_{\beta \in \partial B} \|\zeta - \beta\| \leq h, \quad \forall \zeta \in \partial^+(B \cap \mathbb{Z}_h^2).$$

For any  $z \in \partial^+ B^h$ , there exists  $\zeta \in N(z)$  such that  $\zeta \notin B^h$ . Since  $B^h$  is the discrete interior of  $B \cap \mathbb{Z}_h^2$ , we have  $\zeta \in \partial^+(B \cap \mathbb{Z}_h^2)$ , which implies

$$\min_{\beta \in \partial B} \|z - \beta\| \leq \|z - \zeta\| + \min_{\beta \in \partial B} \|\zeta - \beta\| \leq 2h.$$

(b) case 2:  $z \in \partial^- B^h$ .

By assumption we have  $B^h = (B \cap \mathbb{Z}_h^2)^\circ$  so that

$$z \in \partial^+(B \cap \mathbb{Z}_h^2).$$

As shown above in case 1, we deduce that

$$\min_{\beta \in \partial B} \|z - \beta\| \leq h.$$

(iv) Finally, we come to prove

$$\lim_{h \rightarrow 0^+} \max_{\alpha \in \partial B} \min_{\beta \in \partial B^h} \|\alpha - \beta\| = 0.$$

Since  $\partial B$  is compact, it is sufficient to show that for any  $\epsilon > 0$ , there exists  $\delta > 0$  such that for any  $h < \delta$  and  $z \in \partial B$  we have

$$\min_{\beta \in \partial B^h} \|z - \beta\| < \epsilon,$$

To verify this fact, we let  $\epsilon > 0$  and  $z \in \partial B$  be given. As shown in (7.2), there exists  $\eta > 0$  such that when  $h < \eta$  we have

$$B(z, \epsilon) \cap B^h \neq \emptyset.$$

On the other hand, since  $B(z, \epsilon) \setminus \overline{B}$  is open, there exist  $\eta' > 0$  such that

$$B(z, \epsilon) \cap (\mathbb{Z}_h^2 \setminus \overline{B}) \neq \emptyset, \quad \forall h < \eta'.$$

Thus if  $h < \min\{\eta, \eta'\}$ , one can find a sequence

$$\{z_k\}_{k=1}^n \in \mathbb{Z}_h^2 \cap B(z, \epsilon)$$

with the following properties:

$$z_1 \in B^h, \quad z_n \in \mathbb{Z}_h^2 \setminus B^h, \quad \|z_k - z_{k+1}\| = h \quad (1 \leq k \leq n-1).$$

Now we take

$$k_0 = \max\{k : z_k \in B^h\},$$

which means

$$z_{k_0} \in B^h, \quad z_{k_0+1} \notin B^h.$$

According to the definition of discrete boundary, we have  $z_{k_0} \in \partial B^h$ , which implies that

$$\min_{\beta \in \partial B^h} \|z - \beta\| < \epsilon, \quad \forall h < \min\{\eta, \eta'\}.$$

This completes the proof.  $\square$

**Lemma 7.3.** *Let  $B$  be a bounded open set in  $\mathbb{C}$  and  $f \in C^3(\overline{B}) \cap H(B)$ . If  $B^h \subset B$ , then we have*

$$\max_{B^h} |\partial_{\bar{z}}^h f| = O(h^2).$$

*Proof.* Since  $f$  is holomorphic, we have

$$\partial_{\bar{z}}^h f(z) = \frac{f(z+h) - f(z-h) - \partial_x f(z)2h}{2h} + i \frac{f(z+ih) - f(z-ih) - \partial_y f(z)2h}{2h}.$$

The conclusion follows obviously from the mean value theorem.  $\square$

**Lemma 7.4.** *Let  $B$  be a bounded open set in  $\mathbb{C}$ . If  $B^h$  converges to  $B$ , then we have*

$$\int_{B^h} 1 dV^h = O(1).$$

*Proof.* Since  $B$  is bounded we may assume  $B \subset\subset B(0, R)$ . Therefore, when  $h$  is small enough, we have  $B^h \subset B(0, R)$  so that

$$\int_{B^h} 1 dV^h \leq \int_{\mathbb{Z}_h^2} \chi_{B(0, R)} dV^h.$$

According to the definition of the Haar measure  $V^h$ , the last integral is identical to a Riemann sum, so that

$$\lim_{h \rightarrow 0^+} \int_{\mathbb{Z}_h^2} \chi_{B(0, R)} dV^h = \pi R^2.$$

Therefore,

$$\int_{B^h} 1 dV^h = O(1)$$

as desired.  $\square$

**Lemma 7.5.** *Function  $E$  in (4.1) belongs to  $L^3(\mathbb{Z}^2)$ .*

*Proof.* Recall that

$$E(x, y) = \frac{1}{4\pi^2} \int_{[-\pi, \pi]^2} \frac{2}{i \sin u - \sin v} e^{i(ux+vy)} du dv.$$

Namely,  $E$  is the Fourier transform of function  $1/(i \sin u - \sin v)$  up to a constant factor. The Hausdorff-Young inequality thus tells us that it is sufficient to show

$$\frac{1}{i \sin u - \sin v} \in L^{3/2}(\mathbb{T}^2).$$

Its singular points are given by the solutions of equations

$$\sin u = \sin v = 0,$$

or rather

$$(u, v) \in \{0, \pi, -\pi\}^2.$$

By periodicity, we only need to consider the singular point  $(0, 0)$ . At this point we have

$$\frac{1}{|i \sin u - \sin v|} \simeq \frac{1}{\sqrt{u^2 + v^2}}.$$

It is evident that

$$\frac{1}{\sqrt{u^2 + v^2}} \in L^{3/2}(\mathbb{T}^2),$$

since

$$\int_{[-\pi, \pi]^2} \left( \frac{1}{\sqrt{u^2 + v^2}} \right)^{3/2} dudv < \infty.$$

This completes the proof.  $\square$

Next we show the integrability of the discrete derivatives of  $E$  up to second order.

**Lemma 7.6.** *The discrete derivatives of  $E$  obey the integrability:*

$$\partial_z^1 E \in L^2(\mathbb{Z}^2), \quad (\partial_z^1)^2 E \in L^1(\mathbb{Z}^2).$$

*Proof.* By definition,

$$E(x, y) = \frac{1}{4\pi^2} \int_{[-\pi, \pi]^2} \frac{2}{i \sin u - \sin v} e^{i(ux+vy)} dudv.$$

A direct calculation shows that the discrete derivatives of  $E$  has integral representations

$$(7.3) \quad \partial_z^1 E(x, y) = \frac{1}{4\pi^2} \int_{[-\pi, \pi]^2} \frac{i \sin u + \sin v}{i \sin u - \sin v} e^{i(ux+vy)} dudv$$

and

$$(7.4) \quad (\partial_z^1)^2 E(x, y) = \frac{1}{8\pi^2} \int_{[-\pi, \pi]^2} \frac{(i \sin u + \sin v)^2}{i \sin u - \sin v} e^{i(ux+vy)} dudv.$$

From (7.3), we find that up to a constant  $\partial_z^1 E(x, y)$  is the Fourier transform of the function

$$\frac{i \sin u + \sin v}{i \sin u - \sin v} \in L^2(\mathbb{T}^2).$$

The Hausdorff-Young inequality yields

$$\partial_z^1 E \in L^2(\mathbb{Z}^2).$$

Similarly, (7.4) shows that up to a constant  $\partial_z^2 E(x, y)$  is the Fourier transform of the function

$$\frac{(i \sin u + \sin v)^2}{i \sin u - \sin v}.$$

It is easy to see that the last function, together with its derivatives up to second order, belongs to  $L^{3/2}(\mathbb{T}^2)$ . Again by the Hausdorff-Young inequality, this implies that

$$(1 + x^2 + y^2)(\partial_z^1)^2 E \in L^3(\mathbb{Z}^2).$$

Since  $(1+x^2+y^2)^{-1} \in L^{3/2}(\mathbb{Z}^2)$ , the Hölder inequality shows that  $(\partial_z^1)^2 E \in L^1(\mathbb{Z}^2)$ .

This completes the proof.  $\square$

Theorem 4.5 is about the holomorphicity of the Bochner-Martinelli kernel. We finally give its proof.

Proof of Theorem 4.5. By definition, we have

$$\begin{aligned} \partial_{\bar{\zeta}}^h \mathcal{K}^h(z, \zeta) &= \partial_{\bar{\zeta}}^h A(z - \zeta) n_1^-(z) + \partial_{\bar{\zeta}}^h B(z - \zeta) n_1^+(z) \\ &\quad + i \partial_{\bar{\zeta}}^h C(z - \zeta) n_2^-(z) + i \partial_{\bar{\zeta}}^h D(z - \zeta) n_2^+(z). \end{aligned}$$

We now calculate each summand on the left. By direct calculation, we get

$$\begin{aligned} \partial_{\bar{\zeta}}^h A(z - \zeta) n_1^-(z) &= -4^{-1} \partial_{\bar{\zeta}}^h E(h - z + \zeta) n_1^-(z) \\ &= -4^{-1} \delta_0^h(h - z + \zeta) n_1^-(z). \end{aligned}$$

Similarly,

$$\begin{aligned} \partial_{\bar{\zeta}}^h B(z - \zeta) n_1^+(z) &= -4^{-1} \partial_{\bar{\zeta}}^h E(-h - z + \zeta) n_1^+(z) \\ &= -4^{-1} \delta_0^h(-h - z + \zeta) n_1^+(z) \end{aligned}$$

$$\begin{aligned} \partial_{\bar{\zeta}}^h C(z - \zeta) n_2^-(z) &= -4^{-1} \partial_{\bar{\zeta}}^h E(hi - z + \zeta) n_2^-(z) \\ &= -4^{-1} \delta_0^h(hi - z + \zeta) n_2^-(z) \end{aligned}$$

and

$$\begin{aligned} \partial_{\bar{\zeta}}^h D(z - \zeta) n_2^+(z) &= -4^{-1} \partial_{\bar{\zeta}}^h E(-hi - z + \zeta) n_2^+(z) \\ &= -4^{-1} \delta_0^h(-hi - z + \zeta) n_2^+(z). \end{aligned}$$

Combining the above results to yield

$$\begin{aligned} -4 \partial_{\bar{\zeta}}^h \mathcal{K}^h(z, \zeta) &= \delta_0^h(h - z + \zeta) n_1^-(z) + \delta_0^h(-h - z + \zeta) n_1^+(z) \\ &\quad + i \delta_0^h(hi - z + \zeta) n_2^-(z) + i \delta_0^h(-hi - z + \zeta) n_2^+(z). \end{aligned}$$

It follows immediately that

$$\partial_{\bar{\zeta}}^h \mathcal{K}^h(z, \zeta) = 0 \quad \forall \quad \zeta \notin N(z).$$

since  $\delta_0^h(h - z + \zeta)$ ,  $\delta_0^h(hi - z + \zeta)$  and  $\delta_0^h(-hi - z + \zeta)$  all vanish whenever  $\zeta \notin N(z)$ .

We now come to show that

$$\partial_{\zeta}^h \mathcal{K}^h(z, \zeta) = 0 \text{ for any } \zeta \notin \partial B.$$

We only need to show the first item in the right side vanishes, i.e.

$$\delta_0^h(h - z + \zeta)n_1^-(z) = 0, \quad \forall \zeta \in \mathbb{Z}_h^2 \setminus \partial B, \quad \forall z \in \partial B,$$

since the remaining items also vanishes similarly.

We may assume  $n_1^-(z) \neq 0$ , otherwise there is nothing to prove. In this case, we have

$$\partial_1^{-,h} \chi_B(z) \neq 0.$$

That is,

$$\chi_B(z) - \chi_B(z - h) \neq 0.$$

This implies that either  $z \in B$ ,  $z - h \notin B$  or  $z \notin B$ ,  $z - h \in B$ . In both cases, we have both  $z$  and  $z - h$  are in the boundary  $\partial B$  so that  $\delta_0^h(h - z + \zeta) = 0$  for any  $\zeta \notin \partial B$ . Hence the first item vanishes.  $\square$

## REFERENCES

1. M. Alexa, M. Wardetzky, Discrete Laplacians on general polygonal meshes. *ACM Trans. Graph.* 30(4) (2011), 102:1-102:10.
2. A. I. Bobenko, C. Mercat, Y. B. Suris, Linear and nonlinear theories of discrete analytic functions. Integrable structure and isomonodromic Green's function, *J. Reine Angew. Math.* 583 (2005), 117-161.
3. F. Brackx, H. De Schepper, F. Sommen, L. Van de Voorde, Discrete Clifford analysis: a germ of function theory. *Hypercomplex analysis*, 37-53, Trends Math., Birkhäuser Verlag, Basel, 2009.
4. D. Chelkak, S. Smirnov, Discrete complex analysis on isoradial graphs. *Adv. Math.* 228 (2011), 1590-1630.
5. D. Chelkak, S. Smirnov, Universality in the 2D Ising model and conformal invariance of fermionic observables. *Invent. Math.* 189(3) (2012), 515-580.
6. R. Courant, K. Friedrichs, H. Lewy, Über die partiellen Differenzengleichungen der mathematischen Physik. *Math. Ann.* 100 (1928), 32-74.
7. R. J. Duffin, Basic properties of discrete analytic functions. *Duke Math. J.* 23 (1956), 335-363.
8. J. Ferrand. Fonctions préharmoniques et fonctions préholomorphes. *Bull. Sci. Math.* 68 (1944), 152-180.
9. K. Hildebrandt, K. Polthier, M. Wardetzky, On the convergence of metric and geometric properties of polyhedral surfaces. *Geom. Dedicata* 123 (2006), 89-112.
10. C. Hongler, S. Smirnov, The energy density in the planar Ising model. *Acta Math.* 211 (2013), 191-225.
11. R. Isaacs, A finite difference function theory. *Univ. Nac. Tucumán. Revista A.* 2 (1941), 177-201.

12. R. Kenyon, Conformal invariance of domino tiling, *Ann. Probab.* 28(2) (2000), 759-795.
13. C. Mercat, Discrete Riemann surfaces and the Ising model. *Comm. Math. Phys.* 218 (2001), 177-216.
14. M. Prasolov, M. Skopenkov, Tilings by rectangles and alternating current, *J. Combin. Theory Ser. A* 118(3) (2011), 20-937.
15. M. Skopenkov, The boundary value problem for discrete analytic functions. *Adv. Math.* 240 (2013), 61-87.
16. K. Stephenson, Introduction to circle packing. The theory of discrete analytic functions. Cambridge University Press, Cambridge, 2005.
17. S. Smirnov, Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model. *Ann. Math.* 172 (2010), 1435-1467
18. S. Smirnov, Discrete complex analysis and probability. Proceedings of the International Congress of Mathematicians. Volume I, 595-621, Hindustan Book Agency, New Delhi, 2010.
19. M. Wardetzky, S. Mathur, F. Käberer, E. Grinspun, Discrete Laplace operators: no free lunch, in: A. Belyaev, M. Garland (Eds.), *Eurographics Symp. Geom. Processing*, 2007.
20. D. Zeilberger, Discrete analytic functions of exponential growth. *Trans. Amer. Math. Soc.* 226 (1977), 181-189.

GUANGBIN REN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI 230026, CHINA  
*E-mail address:* `rengb@ustc.edu.cn`

ZEPING ZHU, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI 230026, CHINA  
*E-mail address:* `zzp@mail.ustc.edu.cn`